

# The Klein-Gordon and the Dirac oscillators in a noncommutative space

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## ABSTRACT

We study the Dirac and the Klein-Gordon oscillators in a noncommutative space. It is shown that the Klein-Gordon oscillator in a noncommutative space has a similar behaviour to the dynamics of a particle in a commutative space and in a constant magnetic field. The Dirac oscillator in a noncommutative space has a similar equation to the equation of motion for a relativistic fermion in a commutative space and in a magnetic field, however a new exotic term appears, which implies that a charged fermion in a noncommutative space has an electric dipole moment.

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## I. Introduction

In the last few years theories in noncommutative space have been studied extensively (for a review see Ref. [1]). Noncommutative field theories are related to M-theory compactification [2], string theory in nontrivial backgrounds [3] and quantum Hall effect [4]. Inclusion of noncommutativity in quantum field theory can be achieved in two different ways: via Moyal  $\star$ -product on the space of ordinary functions, or defining the field theory on a coordinate operator space which is intrinsically noncommutative[1,5]. The equivalence between the two approaches has been nicely described in Ref. [6]. A simple insight on the role of noncommutativity in field theory can be obtained by studying the one particle sector, which prompted an interest in the study of noncommutative quantum mechanics (NCQM) [7,8,9,10,11,12,13,14]. In these studies some attention was paid to two-dimensional NCQM and its relation to the Landau problem. It has been shown that the equation of motion of a harmonic oscillator in a noncommutative space is similar to the equation of motion of a particle in a constant magnetic field and in the lowest Landau level [13]. We generalize these relations to the relativistic quantum mechanics. In particular, it is shown that the Dirac and Klein-Gordon oscillators in a noncommutative space have similar behaviour to the Landau problem in a commutative space although an exact map does not exist. However, for the Dirac oscillator there is a new term which is spin dependent. The noncommutative spaces can be realized as spaces where coordinate operator  $\hat{x}^\mu$  satisfies the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \quad (1)$$

where  $\theta^{\mu\nu}$  is an antisymmetric tensor and is of space dimension (length)<sup>2</sup>. We note that space-time noncommutativity,  $\theta^{0i} \neq 0$ , may lead to some problems with unitarity and causality. Such problems do not occur for the quantum mechanics on a noncommutative space with a usual commutative time coordinate. The noncommutative models specified by Eq. (1) can be realized in terms of a  $\star$ -product: the commutative algebra of functions with the usual product  $f(x)g(x)$  is replaced by the  $\star$ -product Moyal algebra:

$$(f \star g)(x) = \exp \left[ \frac{i}{2} \theta_{\mu\nu} \partial_{x_\mu} \partial_{y_\nu} \right] f(x)g(y)|_{x=y} \quad (2)$$

In the case when  $[\hat{p}_i, \hat{p}_j] = 0$ , the noncommutative quantum mechanics

$$H(\hat{p}, \hat{x}) \star \psi(\hat{x}) = E\psi(\hat{x}) \quad (3)$$

reduces to the usual one described by [7-14],

$$H(\hat{p}, \hat{x})\psi(x) = E\psi(x) \quad (4)$$

where

$$\hat{x}_i = x_i - \frac{1}{2\hbar} \theta_{ij} p_j, \quad \hat{p}_i = p_i \quad (5)$$

The new variables satisfy the usual canonical commutation relations:

$$[x_i, x_j] = 0, [p_i, p_j] = 0, [x_i, p_j] = i\hbar\delta_{ij} \quad (6)$$

This paper is organized as follows: in section 2, the Klein-Gordon oscillator in a noncommutative space is investigated and its map to the Landau problem in a commutative space is given. In section 3, the Dirac oscillator in a noncommutative space is defined and its relation to the Landau problem is clarified.

## II. The Klein-Gordon oscillator in a noncommutative space

It is known that the nonrelativistic harmonic oscillator in a noncommutative space has a similar behaviour to the Landau problem in a commutative space [10,11,13,14,20]. In this section we investigate this relation in the relativistic case. The Klein-Gordon oscillator in a two dimensional (2+1 dimensional space-time) commutative space can be defined by the following equation

$$c^2(\vec{p} + imw\vec{r}) \cdot (\vec{p} - imw\vec{r})\psi = (E^2 - m^2c^4)\psi \quad (7)$$

and the energy eigenvalues are given by

$$E_{n_x n_y}^2 = 2mc^2\hbar w(n_x + n_y + 1) + m^2c^4 - 2mc^2\hbar w \quad (8)$$

The nonrelativistic limit of the Eq. (7) is given by

$$[\frac{p^2}{2m} + \frac{1}{2}mw^2r^2]\psi = (\varepsilon + \hbar w)\psi, \quad \varepsilon = E - mc^2 \quad (9)$$

which justifies the name given to it. In a noncommutative space one may describe the Klein-Gordon oscillator by the following equation

$$c^2[(\vec{p} + imw\vec{r}) \cdot (\vec{p} - imw\vec{r})] \star \psi = (E^2 - m^2c^4)\psi \quad (10)$$

which is equivalent to the equation below in a commutative space ( $\theta_{ij} = \epsilon_{ijk}\theta_k$ )

$$c^2[\vec{p} + imw(\vec{r} + \frac{\vec{\theta} \times \vec{p}}{2\hbar})] \cdot [\vec{p} - imw(\vec{r} + \frac{\vec{\theta} \times \vec{p}}{2\hbar})] \psi = (E^2 - m^2c^4)\psi \quad (11)$$

If we put  $\theta_3 = \theta$  and choose the rest of the  $\theta$ -components equal to zero (which can be done by a rotation or a redefinition of coordinates), after straightforward calculation, we arrive at the following equation which can be solved exactly

$$c^2[(1 + \frac{m^2w^2\theta^2}{4\hbar^2})(p_x^2 + p_y^2) + m^2w^2(x^2 + y^2) - \frac{m^2w^2c^2\theta}{\hbar}L_z]\psi = (E^2 - m^2c^4 + 2mc^2\hbar w)\psi \quad (12)$$

and the energy eigenvalues are given by

$$E_{n_x n_y m_\ell}^2 = 2mc^2\hbar w_1(n_x + n_y + 1) - (\frac{m^2w^2c^2\theta}{\hbar})m_\ell\hbar + m^2c^4 - 2mc^2\hbar w \quad (13)$$

where

$$w_1 = w \sqrt{1 + \frac{m^2 w^2 \theta^2}{4\hbar^2}} \quad (14)$$

The energy eigenvalues indicate a similarity to the *normal Zeeman effect*. The nonrelativistic limit of Eq. (12) is given by

$$\left[ \left( \frac{1}{2m} + \frac{mw^2\theta^2}{8\hbar^2} \right) (p_x^2 + p_y^2) + \frac{mw^2(x^2 + y^2)}{2} - \frac{mw^2\theta}{2\hbar} L_z \right] \psi = (\varepsilon + \hbar w) \psi, \quad \varepsilon = E - mc^2 \quad (15)$$

The Klein-Gordon equation for a particle in a commutative space and in a constant magnetic field can be written as

$$c^2 \left[ \left( \vec{p} - \frac{e}{c} \vec{A} \right) \cdot \left( \vec{p} - \frac{e}{c} \vec{A} \right) \right] \psi = (W^2 - m^2 c^4) \psi \quad (16)$$

where

$$A = \frac{\vec{B} \times \vec{r}}{2} \quad (17)$$

A straightforward calculation in the Coulomb gauge yields

$$c^2 \left[ (p_x^2 + p_y^2) + \left( \frac{e^2 B^2}{4c^2} \right) (x^2 + y^2) - \frac{eB}{c} L_z \right] \psi = (W^2 - m^2 c^4) \psi \quad (18)$$

which is quite similar to Eq. (12), although an invertible map between the coefficients does not exist. These arguments can be extended to the three dimensional space and similar results can be obtained. The Klein-Gordon oscillator in a three dimensional (3+1) commutative space can be defined by the following equation

$$c^2 (\vec{p} + imw\vec{r}) \cdot (\vec{p} - imw\vec{r}) \psi = (E^2 - m^2 c^4) \psi \quad (19)$$

and the energy eigenvalues are given by

$$E_{n_x n_y n_z}^2 = 2mc^2 \hbar w (n_x + n_y + n_z + \frac{3}{2}) + m^2 c^4 - 3mc^2 \hbar w \quad (20)$$

In three dimensions (3+1), the Klein-Gordon oscillator in a noncommutative space is given by

$$c^2 [(\vec{p} + imw\vec{r}) \cdot (\vec{p} - imw\vec{r})] \star \psi = (E^2 - m^2 c^4) \psi \quad (21)$$

which is equivalent to the following equation in a commutative space

$$c^2 \left[ \vec{p} + imw \left( \vec{r} + \frac{\vec{\theta} \times \vec{p}}{2\hbar} \right) \right] \cdot \left[ \vec{p} - imw \left( \vec{r} + \frac{\vec{\theta} \times \vec{p}}{2\hbar} \right) \right] \psi = (E^2 - m^2 c^4) \psi \quad (22)$$

After straightforward calculation, we arrive at the following equation ( $\vec{\theta} = \theta \hat{k}$ )

$$\begin{aligned}
& c^2[(1 + \frac{m^2 w^2 \theta^2}{4\hbar^2})(p_x^2 + p_y^2) + m^2 w^2(x^2 + y^2) - \frac{m^2 w^2 \theta}{\hbar} L_z]\psi \\
& = [(E^2 - m^2 c^4) - c^2(p_z^2 + m^2 w^2 z^2 - 3m\hbar w)]\psi
\end{aligned} \tag{23}$$

The above equation can be solved exactly and has a similar behaviour to the dynamics of a scalar particle in a constant magnetic field which is in the z direction and has a coupling with an oscillator in the z direction. It should be noted that a noncommutative space is not isotropic and the energy eigenvalues are given by

$$E_{n_x n_y n_z m_\ell}^2 = 2mc^2 \hbar w_1 (n_x + n_y + 1) + 2mc^2 \hbar w (n_z + \frac{1}{2}) - (\frac{m^2 w^2 \theta}{\hbar}) m_\ell \hbar + m^2 c^4 - 3mc^2 \hbar w \tag{24}$$

where

$$w_1 = w \sqrt{1 + \frac{m^2 w^2 \theta^2}{4\hbar^2}} \tag{25}$$

It should be noted that the following map between the old and new parameters in Eqs. (12) and (18) does not exist.

$$c^2(1 + \frac{m^2 w^2 \theta^2}{4\hbar^2}) \longrightarrow c_n^2 \tag{26}$$

$$c^2 m^2 w^2 \longrightarrow \frac{e_n^2 B_n^2}{4} \tag{27}$$

$$\frac{c^2 m^2 w^2 \theta}{\hbar} \longrightarrow e_n c_n B_n \tag{28}$$

### III. The Dirac oscillator in a noncommutative space

The Dirac oscillator in a commutative space is defined by the following substitution suggested by Ito et al. [15] see also [16,17,18,19],

$$p_i \rightarrow p_i - i\beta m w x_i \tag{29}$$

$$[c\vec{\alpha} \cdot (\vec{p} - i\beta m w \vec{r}) + \beta m c^2]\psi(\vec{r}) = E\psi(\vec{r}) \tag{30}$$

where

$$\psi(\vec{r}) = \begin{pmatrix} \psi_a(\vec{r}) \\ \psi_b(\vec{r}) \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{31}$$

A straightforward calculation leads to the following two simultaneous equations

$$c\vec{\sigma} \cdot (\vec{p} + imw\vec{r})\psi_b(\vec{r}) + mc^2\psi_a(\vec{r}) = E\psi_a(\vec{r}) \quad (32)$$

$$c\vec{\sigma} \cdot (\vec{p} - imw\vec{r})\psi_a(\vec{r}) - mc^2\psi_b(\vec{r}) = E\psi_b(\vec{r}) \quad (33)$$

In Eq. (9),  $\psi_b(\vec{r})$  is the small component of the wave function, which tends to zero in the non-relativistic limit. After some simple rearrangement and use of familiar properties of the spin matrices we find

$$c^2[p_x^2 + p_y^2 + m^2w^2(x^2 + y^2)]\psi_a = [(E^2 - m^2c^4) - c^2(p_z^2 + m^2w^2z^2 - \frac{4mw}{\hbar}\vec{S} \cdot \vec{L} - 3m\hbar w)]\psi_a \quad (34)$$

and one may calculate the energy eigenvalues

$$E^2 = m^2c^4 + (2n + l - 2j + 1)\hbar wmc^2 \quad j = l + \frac{1}{2} \quad (35)$$

$$E^2 = m^2c^4 + (2n + l + 2j + 3)\hbar wmc^2 \quad j = l - \frac{1}{2} \quad (36)$$

The Dirac oscillator in a noncommutative space is given by the following equation,

$$[c\vec{\alpha} \cdot (\vec{p} - i\beta mw\vec{r}) + \beta mc^2] \star \psi(\vec{r}) = E\psi(\vec{r}) \quad (37)$$

Using the new coordinates (5) in a commutative space, we can map the Dirac oscillator in a noncommutative space to a commutative one,

$$[c\vec{\alpha} \cdot (\vec{p} - i\beta mw(\vec{r} + \frac{\vec{\theta} \times \vec{p}}{2\hbar})) + \beta mc^2]\psi(\vec{r}) = E\psi(\vec{r}) \quad (38)$$

The final result for the  $\psi_a$  component is given by

$$c^2[(1 + \frac{m^2w^2\theta^2}{4\hbar^2})(p_x^2 + p_y^2) + m^2w^2(x^2 + y^2) - \frac{m^2w^2\theta}{\hbar}(L_z + 2S_z)]\psi_a =$$

$$[(E^2 - m^2c^4) - c^2(p_z^2 + m^2w^2z^2 - \frac{4mw}{\hbar}\vec{S} \cdot \vec{L} - 3m\hbar w) - \frac{2mwc^2}{\hbar^2}(\vec{S} \times \vec{p}) \cdot (\vec{\theta} \times \vec{p})]\psi_a \quad (39)$$

where  $\theta_{ij} = \epsilon_{ijk}\theta_k$  and  $\theta_k = (0, 0, \theta)$ .

The result shows that the Dirac oscillator in a noncommutative space has a similar behaviour to the Dirac equation in a commutative space describing motion of a fermion in a magnetic field along the z-axis, although an exact map between the parameters does not exist. The other terms are the same as commutative space, i.e. a constant term, an oscillator in the z direction and a spin-orbit coupling term; however, a new interaction term appears which depends on the noncommutativity parameter  $\theta$ , spin and also on the momentum operator. The last term has also an interpretation. For a charged particle in a noncommutative space an electric dipole moment appears [22] which is proportional to

$\vec{\mu}_e \propto \vec{\theta} \times \vec{p}$ . The last term is similar to the spin orbit coupling in a Hydrogen atom Hamiltonian and can be interpreted as a self interaction. A moving particle with an electric dipole causes a magnetic field which self-interacts with magnetic moment  $\vec{\mu}_m = (\frac{e}{mc})\vec{S}$  of the particle

$$\begin{aligned} (\vec{\theta} \times \vec{p}) \cdot (\vec{S} \times \vec{p}) &\propto \vec{\mu}_e \cdot (\vec{S} \times \vec{p}) \\ &\propto \vec{\mu}_m \cdot (\vec{\mu}_e \times \vec{p}) \end{aligned} \quad (40)$$

It is interesting that without any field theory calculations, the Dirac oscillator in a non-commutative space implies an electric dipole moment for a charged particle. The energy eigenvalues can be calculated exactly if we do not consider the last term. The eigenvalues are given by

$$\begin{aligned} E_{n_x n_y n_z m_\ell m_s}^2 &= 2mc^2 \hbar w_1 (n_x + n_y + 1) + 2mc^2 \hbar w (n_z + \frac{1}{2}) - m^2 w^2 c^2 \theta (m_\ell + 2m_s) \\ &\quad - 2mc^2 \hbar w [j(j+1) - \ell(\ell+1) - s(s+1)] + m^2 c^4 - 3mc^2 \hbar w \end{aligned} \quad (41)$$

The energy eigenvalues indicate a similarity to the *anomalous Zeeman effect*.

In two dimensions (2 + 1), this problem is exactly solvable. The same procedure can be applied for the Dirac equation in 2+1 dimensions. In this case the Dirac oscillator can be written as

$$[c\vec{\alpha} \cdot (\vec{p} - imw\beta\vec{r}) + \beta mc^2]\psi = E\psi \quad (42)$$

where  $\vec{\alpha}$  and  $\beta$  are the Pauli matrices and the wave function is a two component spinor  $(\psi_A, \psi_B)$ .

$$\alpha_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \alpha_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (43)$$

The two dimensional Dirac equation can be separated to two equations

$$c^2[(p_x^2 + p_y^2) + m^2 w^2 (x^2 + y^2) - \frac{4mw}{\hbar} (\frac{\hbar}{2})L_z - 2m\hbar w]\psi_A = (E^2 - m^2 c^4)\psi_A \quad (44)$$

$$c^2[(p_x^2 + p_y^2) + m^2 w^2 (x^2 + y^2) + \frac{4mw}{\hbar} (\frac{-\hbar}{2})L_z + 2m\hbar w]\psi_B = (E^2 - m^2 c^4)\psi_B \quad (45)$$

These equations are similar to the three dimensional case and is equivalent to a two dimensional relativistic oscillator with additional spin-orbit terms and a constant of energy which has a different sign for particles and antiparticles. It should be noted that in three dimensions the spin-orbit term has a different sign for particles and antiparticles, but in

the case of two dimensions, as the sign of spin also appears in the equation, it has the same sign in Eqs. (44) and (45). In a noncommutative space, the two dimensional Dirac oscillator can be written as

$$c[(p_x - ip_y) + imw[(x - \frac{\theta p_y}{2\hbar}) - i(y + \frac{\theta p_x}{2\hbar})]]\psi_B = (E - mc^2)\psi_A \quad (46)$$

$$c[(p_x + ip_y) - imw[(x - \frac{\theta p_y}{2\hbar}) + i(y + \frac{\theta p_x}{2\hbar})]]\psi_A = (E + mc^2)\psi_B \quad (47)$$

After straightforward calculation we arrive at the following equations for a particle with spin  $\frac{\hbar}{2}$

$$\begin{aligned} c^2[(1 + \frac{m^2 w^2 \theta^2}{4\hbar^2})(p_x^2 + p_y^2) + m^2 w^2(x^2 + y^2) - \frac{m^2 w^2 \theta}{\hbar}(L_z + 2(\frac{\hbar}{2}))]\psi_A = \\ [(E^2 - m^2 c^4) + c^2(2m\hbar w + \frac{4mw}{\hbar}(\frac{\hbar}{2})L_z) - \frac{2mwc^2}{\hbar^2} \theta(\frac{\hbar}{2})(p_x^2 + p_y^2)]\psi_A \end{aligned} \quad (48)$$

and an antiparticle with spin  $\frac{-\hbar}{2}$

$$\begin{aligned} c^2[(1 + \frac{m^2 w^2 \theta^2}{4\hbar^2})(p_x^2 + p_y^2) + m^2 w^2(x^2 + y^2) - \frac{m^2 w^2 \theta}{\hbar}(L_z + 2(\frac{-\hbar}{2}))]\psi_B = \\ [(E^2 - m^2 c^4) - c^2(2m\hbar w + \frac{4mw}{\hbar}(\frac{-\hbar}{2})L_z) + \frac{2mwc^2}{\hbar^2} \theta(\frac{-\hbar}{2})(p_x^2 + p_y^2)]\psi_B \end{aligned} \quad (49)$$

The above equations are similar to the equation of motion for a fermion on a plane and in a constant magnetic field; however, the last term is an additional interaction which appears in this case and is the same as the exotic term in a three dimensional space. In two dimensions, the problem is exactly solvable and the energy eigenvalues for Eq.(48) are given by

$$E_{n_x n_y m_\ell}^2 = 2mc^2 \hbar w_1 (n_x + n_y + 1) - m^2 w^2 \theta (m_\ell \pm 1) - 2mc^2 \hbar w m_\ell + m^2 c^4 \mp 2mc^2 \hbar w \quad (50)$$

where

$$w_1 = 1 \pm \frac{mw\theta}{\hbar} \quad (51)$$

The critical values of  $\theta = \mp \frac{\hbar}{mw}$  can be interpreted as a point which a resonance occurs. Such resonance points are usually appear in dynamics of a particle with spin and in a magnetic field.

#### IV. Conclusion



We conclude that the known similarity between an oscillator in a noncommutative space and a particle in a constant magnetic field [10,11,13,14,20,21] can be extended to a relativistic motion. The problem is exactly solvable in the spinless cases. However, for the particles with spin or for the Dirac Oscillator in a noncommutative space a new term in the Hamiltonian will appear which can be interpreted as a self-interaction for a charged particle with a dipole electric and magnetic moments.

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